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On Uniqueness of Equilibria in the CAPM

(This paper replaces the paper
"Existence and Uniqueness of Equilibria in the CAPM")

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Abstract

In the standard CAPM with a riskless asset we give a sufficient condition for uniqueness. This condition is a joint restriction on the agents' endowments and their preferences which is compatible with non-increasing absolute risk aversion and which is in particular satisfied with constant absolute risk aversion. Moreover in the CAPM without a riskless asset we give an example for multiple equilibria even though all agents have constant absolute risk aversion.

JEL Class.: G10, C62

keywords: CAPM, uniqueness, risk aversion

1. INTRODUCTION

NOWADAYS the classical two-period Capital Asset Pricing Model is one of the cornerstones of modern finance. Developed by *Sharpe* (1964), *Lintner* (1965), and *Mossin* (1966), it is widely used both by practitioners and theoreticians, since it gives us a manageable and attractive way of thinking about risk and required return on a risky investment. Given this successful theory one is forced to ask why the question of uniqueness of equilibria was not intensively investigated for a long time. It should be clear that without uniqueness the CAPM loses much of its relevance: if there are many equilibria, on which can investors base their investment decisions? And what is the "correct" risk premium for risky assets? In this paper we give a condition for uniqueness of CAPM-equilibria which is based on the risk aversion and the endowments of the investors. Our condition is compatible with non-increasing absolute risk aversion.

It was *Nielsen* (1988) who showed that the static CAPM equilibrium (even in a very simplified example) need not to be unique. There may be several equilibria, all with identical expected total returns, covariances, utility functions, and initial distributions of cash flows. In every CAPM equilibrium, risk is measured in the same way (via the capital market line) but the risk premium is indeterminate. And even worse as it was recently shown by *Bottazzi et al.* (1998), the situation described by Nielsen is by no means exceptional. For every risky market portfolio there are CAPM economies that have arbitrarily many equilibria. This result shows that one cannot expect a general theorem establishing uniqueness for a broad class of CAPM economies.

Up to now only special cases are known in which the static CAPM possesses a unique equilibrium: for example, if utility functions are quadratic or if investors have expected utility functions with constant absolute risk aversion and returns are jointly normally distributed. *Nielsen* (1988) considered the question of uniqueness but he did not give a result that relies on economic fundamentals, i.e. endowments or utility functions, alone. *Dana* (1999) seems to be the most general result on uniqueness. Unfortunately, her criterion is based on the assumption of an additively separable mean-variance utility function which is difficult to interpret economically. Furthermore, *Dana* assumes the mean-variance utility function to be concave. It can be shown that concavity of the mean-variance utility together with *Dana* (1999) assumption implies non-decreasing absolute risk aversion.

Our purpose is to give a uniqueness result that relies only on economically interpretable assumptions. We show that under non-increasing absolute risk aversion a joint restriction on utility functions and endowments implies uniqueness in the CAPM with a riskless asset. It is a common contention that non-increasing absolute risk aversion is a plausible property of utility functions of investors. For example, experiments (see *Levy* (1964)) as well as empirical studies (see *Dalal & Arshanapalli* (1993)) have shown that constant relative risk aversion is very plausible. This gives evidence for decreasing absolute risk aversion. In particular, our condition is satisfied if all investors exhibit constant absolute risk aversion.

Merton (1973) has developed a continuous time version of the CAPM. Recently *Karatzas*

et al. (1990) obtained a uniqueness result for this continuous time model which is based on the assumption of relative risk aversion being smaller than one. But this condition is not compatible with μ - σ utility functions as was shown by *Löffler* (2000). Hence, the idea of *Karatzas et al.* (1990) cannot be applied to the static CAPM.

Our uniqueness result relies heavily on the existence of a riskless asset. This assumption is far from being trivial. Indeed, we will show in section 4 that even in the case when all investors exhibit constant absolute risk aversion, examples of economies with multiple equilibria can be constructed for the CAPM without a riskless asset. The only condition we know which guarantees uniqueness of equilibria even for the CAPM without a riskless asset has been given in *Allingham* (1991). When the CAPM utility functions are derived from expected utility maximization and when asset returns are normally distributed, then agents do not only exhibit constant absolute risk aversion but represented in a mean-variance diagram indifference curves are parallel straight lines.

It should be noted why uniqueness of CAPM-equilibria cannot be derived as a corollary of the uniqueness results known in the standard general equilibrium model (cf. *Arrow & Hahn* (1971) or *Mas-Colell et al.* (1995)). After all the CAPM can be seen as a general equilibrium model with two goods: mean and variance. However, in some important aspects the CAPM differs from the standard general equilibrium model. One of the 'goods', the variance, is not desired and for the other good there are no non-negativity constraints. In particular, there are no non-negativity constraints on the set of attainable payoffs. Introducing such constraints in the CAPM would rule out the important case of normally distributed returns, and moreover, would conflict with the Tobin separation property, a key property of the CAPM. Moreover, in contrast to the standard differentiable approach to general equilibrium theory *Mas-Colell* (1985), indifference curves in the CAPM intersect with the boundary of the consumption set.

The paper is organized as follows. The next section presents the general CAPM in $L^2(\nu)$, and then derives an equivalent two commodity representation. Section 3 gives the uniqueness result for the CAPM with a riskless asset. Section 4 presents an example of a CAPM-economy without a riskless asset that exhibits multiple equilibria even though all investors exhibit constant absolute risk aversion.

2. THE MODEL

The model is essentially a static two-period Capital Asset Pricing Model in a General Equilibrium framework, as it is presented in most modern Finance textbooks, see e.g. *Duffie* (1988) or more recently *LeRoy & Werner* (2001). There are two dates $t = 0, 1$. At date 1 several possible states of the world s can occur. The uncertainty about these states is common among all agents, and described by a probability space (S, \mathcal{S}, ν) . Consumption takes place only at date 1.

A consumption bundle x is a real-valued measurable function on (S, \mathcal{S}, ν) . For technical reasons, the consumption set will be restricted to the set of measurable functions with finite

variance:

$$L^2(\nu) = \{x \mid E(x^2) < \infty\}$$

where $E(x) := \int_S x(s) d\nu$ is the expected value of x . $L^2(\nu)$ is a Hilbert-space, when endowed with the probability adjusted scalar product $x \cdot y = E(xy) = \int_S x(s)y(s) d\nu$ and with the Norm $\|x\| = \sqrt{E(x^2)}$. The covariance-bilinearform $\text{Cov}(x, y) = E(xy) - E(x)E(y)$ and the variance-operator $\text{Var}(x) = \text{Cov}(x, x)$ are defined as usual.

Endowments of the agents are vectors in $L^2(\nu)$. In order to shift consumption among states, the agents may trade assets at date zero, that is, before uncertainty has resolved. An asset is a contract that promises the delivery of some amount of the good in each state of the world at date one. This amount can be negative in some states, as it is for instance when the payoffs of the securities are normally distributed. It is modeled as well as a vector in $L^2(\nu)$. The set of existing non-redundant assets is denoted by Y . There can be potentially an infinite number of securities, but for notational convenience we will always use the symbol ' \sum ' when summing over Y . In any case Y is supposed to be closed. There are no short sales constraints, such that the agents may buy or sell any amount of these assets that they can afford. The set of possible trades of date-1-consumption is given by the linear combinations of securities in Y , thus by $\langle Y \rangle$, the space spanned by Y . The following portfolio η plays a key role in the CAPM. It is the portfolio which solves the minimization problem

$$\min\{\text{Var}(z) \mid E(z) = 1, z \in \langle Y \rangle\} > 0.$$

This portfolio exist because Y is closed. The following Lemma summarizes two properties of η :

Lemma 1

1. *The vector η is collinear to the orthogonal projection of the riskless asset $\mathbf{1}$ onto $\langle Y \rangle$, where orthogonality is meant with respect to the probability adjusted scalar product.*
2. *For all y in $\langle Y \rangle$: $\text{Cov}(\eta, y) = \text{Var}(\eta)E(y)$*

Proof. 1. Let $\hat{\eta}$ be the projection of $\mathbf{1}$ onto $\langle Y \rangle$.

$$\begin{aligned} \hat{\eta} &= \arg \min \{ \|z - \mathbf{1}\|^2 \mid z \in \langle Y \rangle \} \\ &= \arg \min \{ |E(z^2) - 2E(z) + 1| \mid z \in \langle Y \rangle \} \\ &= \arg \min \{ |\text{Var}(z) + E(z)^2 - 2E(z) + 1| \mid z \in \langle Y \rangle \} \\ &= \arg \min \{ \text{Var}(z) + (E(z) - 1)^2 \mid z \in \langle Y \rangle \}. \end{aligned}$$

Since both $\text{Var}(z) \geq 0$ and $(E(z) - 1)^2 \geq 0$ one can conclude

$$\begin{aligned} \hat{\eta} &= \arg \min \{ \text{Var}(z) \mid E(z) = E(\hat{\eta}), z \in \langle Y \rangle \} \\ &= E(\hat{\eta})\eta. \end{aligned}$$

2. Consider the space spanned by η and y . Clearly, since $y \in \langle Y \rangle$,

$$\eta = \arg \min \{ \text{Var}(z) \mid E(z) = 1, z \in \langle \eta, y \rangle \}.$$

Any $z \in \langle \eta, y \rangle$ is of the form $z = \alpha_1 \eta + \alpha_2 y$ with some $\alpha \in \mathbb{R}^2$. The above minimization can be written as a Lagrange–problem of the following form:

$$\min \alpha^T \Omega \alpha, \quad \text{subject to } \alpha \bar{y} = 1,$$

where $\Omega = \begin{pmatrix} \text{Var}(\eta) & \text{Cov}(\eta, y) \\ \text{Cov}(\eta, y) & \text{Var}(y) \end{pmatrix}$ and $\bar{y} = \begin{pmatrix} 1 \\ E(y) \end{pmatrix}$ are the variance-covariance–matrix and the vector of expected returns, respectively. The solution of this problem computes as $\alpha = \lambda \Omega^{-1} \bar{y}$ with some Lagrange multiplier $\lambda > 0$. Since at the minimum $z = \eta$, it must be that $\alpha_2 = 0$. But this implies

$$-\text{Cov}(\eta, y) + \text{Var}(\eta) E(y) = 0.$$

□

A price p of a payoff–stream x in $\langle Y \rangle$ can be identified by virtue of the Riesz representation theorem as a vector in $\langle Y \rangle$. There are I agents $i = \{1, \dots, I\}$ described by their utility functions u^i and their endowments ω^i . The following assumptions are made:

The preference of every agent i can be represented by a mean–variance utility function, that is $u^i(x) = v^i(\mu(x), \sigma(x))$ for all x in $L^2(\nu)$, with $v^i : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ increasing in $\mu(x) = E(x)$, decreasing in $\sigma(x) = \sqrt{\text{Var}(x)}$, continuously differentiable and strictly quasi-concave.

The endowments are ‘spanned’: for all i , $\omega^i \in \langle Y \rangle$. Equivalently, there exist portfolios $\alpha^i \in \mathbb{R}^J$ such that $\omega^i = \sum_j \alpha_j^i y_j$. Furthermore, for the market portfolio $\omega = \sum_i \omega^i$, it is assumed that $E(\omega) = 1$ and $\text{Var}(\omega) > \text{Var}(\eta)$.

An agent’s maximization problem has the form:

$$\begin{aligned} & \max u^i(x) \\ & \text{subject to } p \cdot x \leq p \cdot \omega^i \text{ and } x \in \langle Y \rangle. \end{aligned} \tag{2.1}$$

The definition of an equilibrium of the asset market is standard:

Definition 1

An equilibrium of the CAPM–economy is an allocation $x^{,i} \in \langle Y \rangle$ for every $i \in I$ and a price $p \in \langle Y \rangle$, such that:*

1. $x^{*,i}$ maximizes (2.1) for all $i \in I$
2. $\sum_I x^{*,i} = \omega$

Note that although in general all agents are allowed not to exhaust their budget, in equilibrium they will have to do so, because market clearing requires demand to be equal to supply.

Existence of equilibria has been proved in the CAPM with a riskless asset under assumptions weaker than those we stated above e.g. by *Nielsen* (1990), *Allingham* (1991), *Dana* (1999) and *Hara* (1998). For the case without a riskless asset under some additional assumptions *Nielsen* (1990), *Allingham* (1991) and *Laitenberg* (1998) give existence results. In any case, if there is an equilibrium, the next proposition shows that it features a lot of powerful properties. Based on these properties we follow *Dana* (1999) and transform the general CAPM into a simple equivalent two commodity equilibrium model. We have taken this part of our paper from *Laitenberg* (1998). To the best of our knowledge the transformation from L^2 to \mathbb{R}^2 has so far only been given for the case of a riskless asset¹.

Proposition 1

In an equilibrium, the following relations hold: for every agent i , there exist $\phi^i \in \mathbb{R}$, $\psi^i \in \mathbb{R}_+$ such that

$$x^{*,i} = \phi^i \eta + \psi^i \omega, \quad (2.2)$$

and $\phi_0, \psi_0 \in \mathbb{R}$ such that:

$$p = \phi_0 \eta + \psi_0 \omega. \quad (2.3)$$

Moreover in equilibrium, it must be $p \cdot \omega < p \cdot \eta$.

Proof. ² 1. In equilibrium the agents consume their whole budget.

This must be so, because in equilibrium markets clear and in the aggregate

$$\sum_i p \cdot x^{*,i} = \sum_i p \cdot \omega^i. \quad (2.4)$$

If any agent had $p \cdot x^{*,i} < p \cdot \omega^i$ and all other agents respect their budget constraint $p \cdot x^{*,j} \leq p \cdot \omega^j$, then (2.5) could not hold.

2. Let x^i be the equilibrium choice of agent i at the price p . Decompose $x^i = y^i + z^i$, where $y^i \in \langle \eta, p \rangle$ and z^i is orthogonal to $\langle \eta, p \rangle$. Since $z^i \perp p$, $p \cdot y^i = p \cdot x^i$ and y^i lies in the budget frontier of agent i . Since $z^i \perp \eta$, $\eta \cdot x^i = \eta \cdot y^i$, but this implies

$$\text{Cov}(\eta, x^i) + E(x^i) = \text{Cov}(\eta, y^i) + E(y^i)$$

and with Lemma 2 part 2. this implies $E(x^i) = E(y^i)$.

Since y^i and z^i are orthogonal, $E(y^i z^i) = 0$, and with $E(z^i) = 0$, it immediately follows that $\text{Cov}(y^i, z^i) = 0$, and this implies

$$\text{Var}(x^i) = \text{Var}(y^i) + \text{Var}(z^i) \geq \text{Var}(y^i).$$

Since x^i is most preferred, it must be

$$0 = \text{Var}(z^i) = E((z^i)^2) - (E(z^i))^2 = E((z^i)^2),$$

¹See however *LeRoy & Werner* (2001) for a similar transformation in a more restrictive setting.

²For the special case of utilities derived from quadratic von Neumann-Morgenstern utility of wealth functions, a similar proof has been given by *Geanakoplos & Shubik* (1990).

This in turn implies $z^i = 0$.

Thus for all agents, there are real numbers λ^i, ξ^i , such that

$$x^i = \lambda^i \eta + \xi^i p.$$

3. In equilibrium $\sum_i x^i = \sum_i \lambda^i \eta + \sum_i \xi^i p = \omega$. Thus

$$p = \frac{1}{\sum_i \xi^i} \omega - \frac{\sum_i \lambda^i}{\sum_i \xi^i} \eta,$$

if $\sum_i \xi^i \neq 0$. The inverse, $\sum_i \xi^i = 0$, is excluded, since it was assumed that ω and η are not collinear.

Writing p in the above form, one concludes:

$$x^i = \left(\lambda^i - \frac{\xi^i}{\sum_i \xi^i} \sum_i \lambda^i \right) \eta + \frac{\xi^i}{\sum_i \xi^i} \omega.$$

4. Assume $p \cdot \omega \geq p \cdot \eta$. Let $x^i = \phi^i \eta + \psi^i \omega$ be the choice of agent i , and assume $\psi^i > 0$. Take $y = (\phi^i + \psi^i) \eta$. Clearly $E(y) = E(x^i)$ and $\text{Var}(y) < \text{Var}(x^i)$. Furthermore $p \cdot y = (p \cdot \eta)(\phi^i + \psi^i) \leq \phi^i(p \cdot \eta) + \psi^i(p \cdot \omega) = p \cdot x^i$. Thus x^i cannot be maximizing agent i 's utility function. Therefore $\psi^i \leq 0$ for all i . But then $\sum_i \psi^i \leq 0$, which is impossible, because at equilibrium $\sum_i x^i = \omega$ and therefore $\sum_i \psi^i = 1$. Thus it must be: $p \cdot \omega < p \cdot \eta$.

5. Assume $x^i = \phi^i \eta + \psi^i \omega$ with $\psi^i < 0$. Then $y = (\phi^i + \psi^i) \eta$ has $E(y) = E(x^i)$ and $\text{Var}(y) < \text{Var}(x^i)$ and since $p \cdot \omega < p \cdot \eta$, $p \cdot y = (p \cdot \eta)(\phi^i - |\psi^i|) < \phi^i(p \cdot \eta) - |\psi^i|(p \cdot \omega) = p \cdot x^i$. Therefore x^i cannot be agent i 's choice, and thus $\psi^i \geq 0$ for all $i \in I$. \square

Given the insights of the last proposition, information about the equilibrium can be gained by solving a simplified problem, namely one in which investors do not choose their portfolio among all securities in Y , but only among η and ω . The investors' new problem then has the following shape:

$$\max u^i(\phi \eta + \psi \omega) \tag{2.5}$$

$$\text{subject to } p \cdot \eta \phi + p \cdot \omega \psi \leq p \cdot \eta \bar{\phi}^i + p \cdot \omega \bar{\psi}^i,$$

where $\bar{\omega}^i = \eta \bar{\phi}^i + \omega \bar{\psi}^i$ is the orthogonal projection of ω^i onto $\langle \eta, \omega \rangle$. $\bar{\phi}^i$ and $\bar{\psi}^i$ can be computed as:

$$\bar{\psi}^i = \frac{\text{Cov}(\eta, \omega^i) - \text{Cov}(\omega, \omega^i)}{\text{Var}(\eta) - \text{Var}(\omega)}, \quad \bar{\phi}^i = E(\omega^i) - \bar{\psi}^i.$$

It will turn out that it is more practical to use instead of the base elements η and ω , the equivalent base consisting of η and $\omega - \eta$. The use of $(\omega - \eta)$ has several advantages. First, for the general perception of the facts, it is very intuitive to have one base element closely related to the entity 'standard deviation'. Secondly, since some computing will have to be done, it is

obviously very helpful to have $E(\omega - \eta) = \text{Cov}(\eta, \omega - \eta) = 0$. Finally, the main reason for the use of $(\omega - \eta)$ as a base element is that due to the last line in proposition 1 in equilibrium its price has to be negative, so that we can use the price normalization $p \cdot (\omega - \eta) = -1$.

From now on we use the following notation. The space of allowed asset-trades is spanned by η and $\hat{\sigma} = \frac{1}{\|\omega - \eta\|^2}(\omega - \eta)$. Given the assumptions on ω , one has:

$$E(\hat{\sigma}) = 0, \quad \text{Var}(\hat{\sigma}) = 1.$$

For a consumption vector $x = \alpha\eta + \beta\hat{\sigma}$, the following holds:

$$E(x) = \alpha, \quad \text{Var}(x) = \text{Var}(\eta)\alpha^2 + \beta^2.$$

The endowment of agent i is given as

$$\bar{\omega}^i = \bar{\alpha}^i\eta + \bar{\beta}^i\hat{\sigma},$$

with $\bar{\alpha}^i = \bar{\phi}^i + \bar{\psi}^i$ and $\bar{\beta}^i = \|\omega - \eta\|^2\bar{\psi}^i$.

An equilibrium in these new notations is then defined as:

Definition 2 *An equilibrium of the $\mu - \sigma$ -economy is an allocation $\alpha^{*,i}, \beta^{*,i} \in \mathbb{R}^2$ for every $i \in I$ and a price $r^* \in \mathbb{R}$, such that:*

1. $\alpha^{*,i}, \beta^{*,i}$ maximizes

$$v^i(\alpha, \sqrt{\text{Var}(\eta)\alpha^2 + \beta^2}) \tag{2.6}$$

$$\text{subject to} \quad r^*\alpha - \beta \leq r^*\bar{\alpha}^i - \bar{\beta}^i,$$

2. $\sum_I \alpha^{*,i} = 1, \quad \sum_I \beta^{*,i} = \|\omega - \eta\|^2.$

This definition concludes the presentation of the model.

3. UNIQUENESS OF EQUILIBRIA IN THE CAPM WITH A RISKLESS ASSET

In this section we assume that there is a riskless asset, i.e. $\mathbf{1} \in \langle Y \rangle$. In this case $\eta = \mathbf{1}$ and the price of η has to be positive as can be shown by a simple no-arbitrage argument, see *Bottazzi et al.* (1998). We can then use the price normalization $p \cdot \eta = 1$. Now the decision problem derived in the previous section can be simplified to

$$\max_{(\mu, \sigma)} v^i(\mu, \sigma)$$

$$\text{s.t. } \mu - q\sigma \leq \bar{\mu}^i - q\bar{\sigma}^i$$

where $\bar{\mu}^i = E(\omega^i)$ and $\bar{\sigma}^i = \sqrt{\text{Cov}(\omega^i, \omega)}$. The discussion of the various conditions for uniqueness will be in terms of the slope of the indifference curves in the (μ, σ) - diagram,

$$S^i(\mu, \sigma) := -\frac{\partial_\sigma v^i(\mu, \sigma)}{\partial_\mu v^i(\mu, \sigma)}.$$

This slope is a measure of the investor's risk aversion, as Nielsen (1988) and Lajari and Nielsen (1994) have shown: Define a simple portfolio problem as a decision problem where an investor maximizes utility over portfolios formed by one riskless and one risky asset. Lajari and Nielsen (1994) then show that non-increasing risk aversion is equivalent to the property that in every simple decision problem the demand of the risky asset is non-increasing in income. Moreover, as we will show, how $S^i(\mu, \sigma)$ changes with μ is the key determinant for the monotonicity of the agent's optimal choice σ^i as a function of q . The following terminology is quite useful.

Definition 3

A mean-variance utility function exhibits non increasing (non decreasing) risk aversion if for all $\sigma \geq 0$ the slope of its indifference curves are non increasing (non decreasing) functions in mean. It exhibits constant absolute risk aversion if this slope is constant.

The next Lemma shows how the slope of an agent's risk aversion $\partial_\mu S^i(\mu^i, \sigma^i)$ determines the slope of his demand for risk $\partial_q \sigma(q)$.

Lemma 2(individual monotonicity)

Let $(\mu^i(q), \sigma^i(q)) \in \arg\max v^i(\mu, \sigma)$ s.t. $\mu - q\sigma \leq \bar{\mu}^i - q\bar{\sigma}^i$. Then

$$(a) \partial_q \sigma^i(q) \gtrless 0 \text{ iff } \partial_\mu S^i(\mu^i(q), \sigma^i(q))(\sigma^i(q) - \bar{\sigma}^i) \lesseqgtr 1 \text{ and}$$

$$(b) \partial_q \sigma^i(q) > 0 \text{ if } \partial_\mu S^i(\mu^i(q), \sigma^i(q))(\mu^i(q) - \bar{\mu}^i(q)) \leq 0.$$

Proof

By assumption higher mean is preferred, hence the budget restriction holds with equality. Therefore, consider

$$\max_{\sigma \geq 0} v^i(\bar{\mu}^i - q\bar{\sigma}^i + q\sigma, \sigma).$$

The first order condition for an optimal solution is
(FOC) $\partial_\mu v^i(\bar{\mu}^i - q\bar{\sigma}^i + q\sigma, \sigma)q + \partial_\sigma v^i(\bar{\mu}^i - q\bar{\sigma}^i + q\sigma, \sigma) = 0.$

Yielding

$$\frac{d\sigma}{dq} = \frac{[(\partial_\mu v^i) + (\partial_\mu^2 v^i)q(\sigma - \bar{\sigma}^i) + (\partial_\mu \partial_\sigma v^i)(\sigma - \bar{\sigma}^i)]}{-[(\partial_\sigma \partial_\mu v^i) + \partial_\mu^2 v^i q + (\partial_\mu \partial_\sigma v^i)q + (\partial_\sigma^2 v^i)]}$$

Note that by strict quasi-concavity of v^i the denominator in this expression is strictly positive. Furthermore, since $(\partial_\mu v^i) > 0$ it is obtained that

$$\frac{d\sigma}{dq} > 0 \text{ iff } 1 + \frac{[(\partial_\mu^2 v^i)q + (\partial_\mu \partial_\sigma v^i)](\sigma - \bar{\sigma}^i)}{(\partial_\mu v^i)} > 0..$$

Substitution of $q = -\frac{\partial_\sigma v^i}{\partial_\mu v^i}$ from the (FOC) and using the identity $\partial_\mu S^i = -\frac{[(\partial_\mu \partial_\sigma v^i)(\partial_\mu v^i) - (\partial_\mu^2 v^i)(\partial_\sigma v^i)]}{(\partial_\mu v^i)^2}$ proves our claim (a).

To see the second claim (b), substitute $(\sigma - \bar{\sigma}^i) = \frac{(\mu - \bar{\mu}^i)}{q}$ from the budget equation into claim (a), recognizing that $q > 0$.

□

From Lemma 3 we get the following condition for uniqueness of equilibria in the CAPM:

Theorem 1

When there is a riskless asset, CAPM equilibria are unique if every agent has on the one hand non-decreasing (on the other hand non-increasing) absolute risk aversion for mean-variance combinations with mean smaller (resp. larger) than the expected value of her endowments and variance smaller (resp. larger) than the covariance of her endowments and the market portfolio.

Proof

For each $q > 0$, every agent $i = 1, \dots, I$ can be in one of the following situations:

	$\partial_\mu S^i$	$\partial_q \sigma^i(q)$	result follows
(1) $\sigma^i(q) < \bar{\sigma}^i, \mu^i(q) < \bar{\mu}^i$	0	≥ 0	from (a) or from (b)
(2) $\sigma^i(q) < \bar{\sigma}^i, \mu^i(q) \geq \bar{\mu}^i$	≤ 0	> 0	from (b)
(3) $\sigma^i(q) \geq \bar{\sigma}^i, \mu^i(q) < \bar{\mu}^i$	≤ 0	> 0	from (a)
(4) $\sigma^i(q) \geq \bar{\sigma}^i, \mu^i(q) \geq \bar{\mu}^i$	≤ 0	> 0	from (a) or from (b)

Thus every individual demand function is strictly monotone and therefore "market demand" $\sum_{i=1}^I \sigma^i(q)$ is strictly monotone. Hence there is at most one equilibrium. From the existence results mentioned above follows that there is at least one equilibrium. Therefore, there is a unique equilibrium.

□

- Figure 1 (see end of text) -

Figure 1 shows a set of indifference curves satisfying the assumptions of Theorem 1. In the area (1), for all σ the slope of the indifference curves are non-decreasing as a function of μ . In areas (2), (3) and (4) for all σ this slope is non-increasing in μ .

The condition used in Theorem 1 is a joint restriction on individual's resources and their preferences. The following Corollary 1 demonstrates uniqueness of equilibria as being derived from an assumption on v^i only.

Corollary 1

CAPM equilibria are unique if all agents have constant risk aversion.

Proof

In the case of constant risk aversion, $\partial_\mu S^i = 0$ for all $i = 1, \dots, I$. Hence both conditions of Lemma 1 can be applied to obtain this result. □

Corollary 1 establishes uniqueness in the CAPM for the case of constant absolute risk aversion without any distributional assumptions on the payoffs thus generalizing the well known uniqueness result for the case of exponential utility functions and normally distributed returns.

An interesting observation, which was pointed out to us by an anonymous referee of this journal, is that constant absolute risk aversion holds if and only if the mean variance utility functions $v^i(\mu, \sigma)$ are quasi-linear, i.e. if they can be represented as $v^i(\mu, \sigma) = \mu - h_i(\sigma)$ for some convex function h_i .³

This observation relates the uniqueness of CAPM-equilibria in this case nicely to the results on uniqueness found in the general equilibrium literature since quasi-linearity is a well known condition for uniqueness of general equilibria. However, without a riskless asset quasi-linearity does not need to follow from constant absolute risk aversion. And indeed in the next section we will demonstrate by means of an example that without a riskless asset CAPM-equilibria need not be unique even though all agents have constant absolute risk aversion.

4. MULTIPLICITY OF EQUILIBRIA IN THE CAPM WITHOUT A RISKLESS ASSET

The framework in this section is the same as in section 3, except that we assume now $\mathbf{1} \notin Y$.

In the following we will demonstrate that condition given in Theorem 1 is not sufficient to guarantee a unique equilibrium in the CAPM with risky assets only. We will construct an

³The "if" part of this claim follows directly from this functional form by applying it to the definition of constant absolute risk aversion, i.e. to $\partial_\mu S^i(\mu, \sigma) = 0$. To see the "only if" part note that constant absolute risk aversion implies that the mean-variance-preference ordering of any two points in L^2 or equivalently in the mean-variance diagram remains unchanged by addition of any scalar multiple of the riskless asset to both points. Hence for any two points $(\mu_1, \sigma_1), (\mu_2, \sigma_2)$ such that $v^i(\mu_1, \sigma_1) = v^i(\mu_2, \sigma_2)$ we get that $v^i(0, \sigma_1) = v^i(\mu_2 - \mu_1, \sigma_2)$. That is to say since the LHS is independent from μ the RHS must also be independent from μ thus there must exist some convex function h_i such that $v^i(\mu, \sigma) = \mu - h_i(\sigma)$.

economy with two assets and two constantly risk averse investors that exhibits (at least) two equilibria.

To make things easier we assume $\text{Var}(\eta) = 1$. Recall from the transformation outlined in section 2 that any investor i solves the maximization problem

$$\max v^i(\alpha, \sqrt{(\alpha)^2 + (\beta)^2}) \quad \text{s.t.} \quad \alpha - q\beta \leq \bar{\alpha}^i - q\bar{\beta}^i. \quad (4.7)$$

This formulation already reveals the intuition of our example. Although the utility function in μ and σ is quasi-linear, the transformed problem in the (α, β) space will not be quasi-linear. Hence, multiple solutions might exist and it remains to show the details. To this end we choose the endowments of the investors as

$$(\bar{\alpha}^1, \bar{\beta}^1) = (0.546, 0.975), \quad (\bar{\alpha}^2, \bar{\beta}^2) = (0.454, 0.025). \quad (4.8)$$

Notice that both endowments add to one which will be important later.

The utility functions exhibit constant absolute risk aversion, hence the utility v^i are of the functional form

$$v^i(\mu, \sigma) = \mu - h_i(\sigma)$$

where h_i is a strictly convex and strictly monotone function. For our example it will be enough to show that the first derivative is positive and monotone at two points since then h_i can easily be extended on \mathbb{R}_+ . In particular we assume that

$$\begin{aligned} h'_1(0.17) &= 1.2 & h'_1(0.172) &= 1.21 \\ h'_2(1.152) &= 0.3718 & h'_2(1.17) &= 0.6667 \end{aligned}$$

To derive the equilibria consider the first order conditions according to (4.7), $i = 1, 2$

$$\begin{aligned} 1 - 2\alpha h'_i(\alpha^2 + \beta^2) - \lambda q &= 0 \\ -2\beta h'_i(\alpha^2 + \beta^2) + \lambda q &= 0. \end{aligned}$$

The budget constraint must be binding at equilibrium if $\lambda_i > 0$ which is implied by $1 - 2\alpha h'_i(\alpha^2 + \beta^2) > 0$. The first order conditions can therefore be simplified to

$$\frac{1 - 2\alpha h'_i(\alpha^2 + \beta^2)}{2\beta h'_i(\alpha^2 + \beta^2)} = q > 0$$

which will ensure non-satiation at equilibrium. If markets clear and since endowments add to one this condition can be written using the optimal portfolio (α^*, β^*) of investor one

$$\frac{1 - 2\alpha^* h'_1(\alpha^{*2} + \beta^{*2})}{2\beta^* h'_1(\alpha^{*2} + \beta^{*2})} = \frac{1 - 2(1 - \alpha^*) h'_2((1 - \alpha^*)^2 + (1 - \beta^*)^2)}{2(1 - \beta^*) h'_2((1 - \alpha^*)^2 + (1 - \beta^*)^2)} = q^* > 0. \quad (4.9)$$

Our example will be complete if we determine the optimal portfolio (α^*, β^*) of investor one and show that $(1 - \alpha^*, 1 - \beta^*)$ is optimal for investor two or, which is the same, satisfies (4.9). The following portfolios and prices will do

$$\begin{aligned} (\alpha_{(1)}^*, \beta_{(1)}^*) &= (0.4, 0.1), & (1 - \alpha_{(1)}^*, 1 - \beta_{(1)}^*) &= (0.6, 0.9), & q_{(1)}^* &= 0.16667, \\ (\alpha_{(2)}^*, \beta_{(2)}^*) &= (0.11, 0.6), & (1 - \alpha_{(2)}^*, 1 - \beta_{(2)}^*) &= (0.89, 0.4), & q_{(2)}^* &= 0.75806. \end{aligned}$$

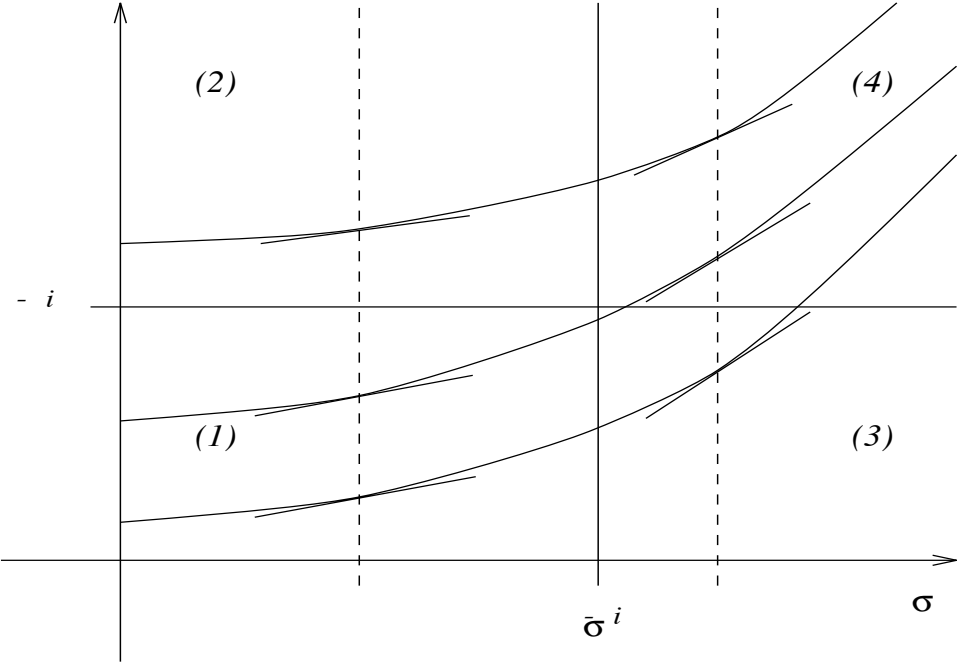
Using our assumption on the first derivative of h_1 and h_2 it is now straightforward to verify that both equilibria satisfy our condition (4.9). Moreover at these equilibria all budget constraints are indeed satisfied with equality. We therefore have constructed a CAPM with at least two equilibria.

Finally note that if the functions $h_i(\sigma)$ were of the functional form $c_i\sigma^2$ for some positive constant c_i , then our construction of multiple equilibria does not work because the single parameter c_i is not sufficient to determine two values of h'_i . And indeed in this case, as *Allingham* (1991) has demonstrated, equilibria are unique even when there is no riskless asset.

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- Figure 1 -